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Recontracting and stochastic stability in cooperative games

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Abstract

An evolutionary style model of recontracting is given which guarantees convergence to core allocations of an underlying cooperative game. Unlike its predecessors in the evolution/learning literature, this is achieved without assumptions of convexity of the characteristic function or a reliance on random errors. The stochastic stability properties of the model are then examined and it is shown that stochastically stable states solve a simple and intuitive minimization problem which reduces to maximizing a Rawlsian SWF for a common class of utility functions. In contrast to previous analyses, the stochastically stable state is unique for a broad class of utility functions.

JEL classification: C71; C72; C73; C78

Keywords: Bargaining; Learning; Core; Convexity; Coalitions

1. Introduction

The papers of Green [1] and Feldman [2] show how a process of coalitional recontracting can converge to core allocations of a cooperative game under specific assumptions. The papers of Agastya [3,4] and Rozen [5] achieve similar results for myopic adaptive processes applied to non-cooperative representations of characteristic function form games: for convex characteristic

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functions the steady states of the processes lie in the core. Framing the model as an adaptive process applied to a non-cooperative game with explicit individual strategies allows the theory of stochastic stability [6,7] to be used to determine which of the steady states of the process is most robust to random errors made by players when choosing their strategies.

This paper unites these two strands of literature by incorporating joint strategic switching into the social learning dynamic which drives strategic change in the non-cooperative representation of any given characteristic function form game. Convergence to interior core allocations is shown for this dynamic under assumptions similar to those of [1], with some of the more restrictive assumptions replaced with additional symmetry and uncertainty in the recontracting process. In contrast to [3,5] and other non-cooperative models of coalition formation [8,9] which assume convexity or total balancedness of the characteristic function, the process in the paper gives convergence for any superadditive characteristic function with nonempty interior core. This result is summarized in Theorem 1.

The second theorem of this paper characterizes the stochastically stable states of the dynamic process – given the players' utility functions $u(.)$, it selects within the core allocations. For a common class of utility functions, those of the form $u(d) = ad^b$, stochastically stable states maximize a Rawlsian social welfare function: the stability of a core allocation increases in the wealth of the poorest player. Furthermore, when $u(.)/u'(.)$ is convex, a class of utilities which includes CRRA utility with preexisting wealth, the stochastically stable state is unique. When this condition holds and $u(.)$ is concave, the stochastically stable state is determined by a trade-off between maximizing the wealth of the poorest player and minimizing wealth inequality amongst the remaining players.

The learning dynamic of this paper involves groups of players jointly adjusting their strategies. Each player's strategy includes a quantity of good demanded and a set of players with whom he is willing to form coalitions. Players do not necessarily form coalitions with those with whom they discuss strategy: sometimes it is in players' best interests to agree not to be part of the same coalition. The model of this paper nests several other models. If the size of a group of players who can jointly adjust their strategies is restricted to equal one, the model is effectively that of [5]. Adding a further restriction that all players are always willing to form coalitions with all other players we get a similar model to [3]. A final restriction that gives zero value to all coalitions other than the grand coalition reduces the model to the model of Nash demand games of Young [10].

In [4] the stochastically stable states minimize the maximum weighted payoff across all players. This is because the player whose best response can change after the fewest random errors is the richest player. The presence of joint strategic switching in the current paper allows the wealth of every player to play a role in determining how vulnerable a state is to random errors and so allows a more precise characterization of stochastically stable states. However, equity considerations do still play a significant role in the selection criterion. This paper also differs from the aforementioned paper in not requiring Inada conditions on the players' utilities.

A related paper is [11] which gives stochastic stability results for coalitional recontracting in a housing economy. In the model of that paper there are multiple goods – houses – and each individual can own one and only one house. The present model has one good which can be held in any quantity. Another paper which gives core convergence is [12] which unlike this paper and the other papers in the literature relies on random errors occurring outside of the core to move the process to a core allocation.

The paper is organized as follows: Section 2 describes the stage game. Section 3 describes the learning dynamic. Section 4 is concerned with convergence to the core of the unperturbed dynamic and contains Theorem 1. Section 5 is concerned with an analysis of the stochastically stable states of our model and contains Theorem 2. Section 6 discusses some of the innovations of this paper and identifies avenues for future work. All proofs are relegated to Appendix A.

2. Model

A game $\Gamma = \{I, v, \delta, F, u\}$ is defined as follows. There are N players indexed $I = \{1, \ldots, N\}$. $\mathcal{I} = \mathcal{P}(I)$ is the set of coalitions: the set of all subsets of I. $v : \mathcal{I} \to \mathbb{Q}$ is called the characteristic function, assumed to satisfy the following:

Assumption 1 *(Superadditivity).* If $S \cap T = \emptyset$, $v(S \cup T) \geqslant v(S) + v(T)$.

Assumption 2 *(Nonempty interior core).*

$$\mathcal{C}^*(v) = \left\{ d \in \mathbb{R}^N \colon \sum_{i \in I} d_i = v(I),\ \sum_{i \in S} d_i > v(S)\ \forall S \subset I \right\} \neq \emptyset$$

The core, $\mathcal{C}(v)$, is defined identically but with the strict inequality replaced by a weak one. The smallest money unit, δ, is such that for all S, $v(S)$ is an integral multiple of δ. Σ_i^δ is defined as the set of all integral multiples of δ that lie in the interval $[v(\{i\}), v(I)]$. δ is assumed small enough that $\{d \in \mathcal{C}^*(v) | d_i \in \Sigma_i^\delta\ \forall i\}$ is nonempty.[2]

This paper looks at a non-cooperative representation of the cooperative game $\{I, v\}$ in which each player i chooses a set of all possible players $P_i \subseteq I$ with whom he is willing to form coalitions. Assume a player is always willing to form a coalition with himself, that is for all i, $i \in P_i$. Players also choose a demand $d_i \in \Sigma_i^\delta$. The strategies chosen by all players are denoted (d, P) where $d = (d_1, \ldots, d_N)$ and $P = (P_1, \ldots, P_N)$. (d, I) is used as shorthand when $P = (I, \ldots, I)$.

Denote the set of partitions of I as $\Pi(I)$.

Definitions. *Demand feasibility* for $\pi \in \Pi(I)$ and strategies (d, P) is satisfied if $\forall S \in \pi$ with $|S| \geqslant 2$, $\sum_{i \in S} d_i \leqslant v(S)$. $\pi \in \Pi(I)$ and (d, P) are *mutually compatible* if $\forall S \in \pi$, $i \in S \Longrightarrow S \subseteq P_i$. A coalition structure $\pi \in \Pi(I)$ is *feasible* for (d, P) if it satisfies mutual compatibility and demand feasibility.

The game proceeds as follows. Players choose strategies simultaneously. $\pi \in \Pi(I)$ is then chosen according to a distribution $F_{(d,P)}(\pi)$ over the coalition structures π which are feasible for (d, P). The family of these distributions indexed by (d, P) is denoted F.

Assumption 3. If π^1 and π^2 are feasible for (d, P) and π^2 is a coarsening of π^1 then $F_{(d,P)}(\pi^1) = 0$.

This restriction simply says that all else being equal, players will form larger coalitions rather than smaller ones. No further conditions are placed on $F_{(d,P)}$. If π denotes a non-singleton coalition for player i then he receives d_i of the good. Otherwise he receives $v(\{i\})$ of the good.

[2] Discretization is carried out to aid stochastic stability analysis and Theorem 2.

The amounts of good received by each player are referred to as an *allocation*. Each player has a strictly increasing vNM utility function $u_i : \mathbb{R}_+ \to \mathbb{R}$ which is a function of the amount of good received. Let $U_i((d, P))$ be the expected utility of player i when strategy profile (d, P) is played.

Allocations in $\mathcal{C}^*(v)$ can only be supported if the grand coalition forms, although the formation of the grand coalition does not guarantee that the ensuing allocation will be in $\mathcal{C}^*(v)$. The grand coalition can only form if $P_i = I \ \forall i \in I$. That is, all players must be willing to form coalitions with all other players for the grand coalition to be able to form. Γ has many Nash equilibria. For example, any profile with $P_i = \{i\} \ \forall i \in I$ is a Nash equilibrium no matter what demands d_i are. Any profile (d, P) with $d \in \mathcal{C}^*(v)$, $P_i = I \ \forall i \in I$ is a strict Nash equilibrium of Γ: if a given player were to increase his demand then no coalitions which include that player would satisfy demand feasibility so he would obtain $v(\{i\})$ of the good. The reverse inclusion does not hold: not all strict Nash equilibria are contained in $\mathcal{C}^*(v)$, as is demonstrated in Example 4.1 later in the paper.

3. Learning

The learning process in this paper features sets of players jointly choosing their actions so as to best respond to the state of the world as they perceive it to be. These players do not form exclusive coalitions but instead jointly decide the set of coalition partners they are each willing to accept which may or may not include one another – players can agree to keep out of each other's way. The dynamic proposed is essentially adaptive play as in [13] with some work going into defining what constitutes a best response for a group of players. Under adaptive play, players repeatedly play an N player game Γ. Each period, strategies are determined before Γ is played. Γ is as described in Section 2.

We note that in [13] adaptive play is interpreted as modeling situations in which each player is a representative agent picked at random from some population of similar agents. This interpretation is also a valid possibility for this paper, but for the sake of clarity we stick to using the term 'player' to describe a position in a game, whether it is the same agent repeatedly playing or various agents plucked randomly with replacement from some underlying population. In order to facilitate a stochastic stability analysis in the second half of this paper we will need the opportunity to perturb the learning process by introducing random errors. When choosing a strategy there is a probability $\varepsilon \geqslant 0$ that any given player will play a strategy chosen at random from the set of all possible strategies (demands and coalition requests). These errors occur independently across players. It is emphasized that Theorem 1 relates to the process without random errors, $\varepsilon = 0$.

3.1. Sampling

Define (d^t, P^t) as the actions played at time t. Let the history $h^t = ((d^t, P^t), (d^{t-1}, P^{t-1}), \ldots, (d^{t-m+1}, P^{t-m+1}))$ denote the action profiles for the $m \in \mathbb{N}$ periods up to time t. Let $\mathcal{H}$ denote the set of all possible histories. The change in actions from period to period can then be modeled as a Markov chain. At the start of each period each player i randomly samples k_i of the previous m periods. Assume $m \geqslant 2 \max_i k_i$.[3] Let the periods in player i's sample be denoted $\sigma_i^t \subset \{t-1, t-2, \ldots, t-m\}$, $|\sigma_i^t| = k_i$. For each period sampled every player's action

[3] The condition that m be 'large enough' is common in the literature.

is observed. Following [4] it is assumed that a player considers each of the d_{-i} from the sampled periods to be equally likely: he assumes that the actions of his opponents are correlated. Let $U_i((d,P)_S|\sigma_i^t)$ denote player i's expected payoff when players in set $S \subseteq I$ take actions $(d,P)_S$ and i assumes the players outside S play with probabilities in proportion to his sample:

$$U_i\big((d,P)_S|\sigma_i^t\big) := \frac{1}{k_i}\sum_{\tau\in\sigma_i^t} U_i\big((d,P)_S,\big(d^\tau,P^\tau\big)_{I\setminus S}\big)$$

The minimum acceptable expected payoff $\underline{u}_i^t$ for a player is defined to be his average payoff from the periods in his sample:

$$\underline{u}_i^t := \frac{1}{k_i}\sum_{\tau\in\sigma_i^t} U_i\big(\big(d^\tau,P^\tau\big)\big)$$

3.2. *Choice of strategy*

Each player independently makes an error with probability $\varepsilon \geqslant 0$. If a player i makes an error he selects a strategy at random according to a distribution with full support over all strategies and we set $\varkappa_i^t = 1$. Otherwise we let $\varkappa_i^t = 0$. $\varkappa^t$ is the matrix of $\varkappa_i^t$ from periods $t-m+1$ to t, $\varkappa^t \in \{0,1\}^{N\times m} =: \mathcal{K}$. Denote the set of players who have not made an error in the current period as $I_{\setminus\varepsilon}^t = \{i \in I\colon \varkappa_i^t = 0\}$. A partition of non-error making players $\rho^t \in \Pi(I_{\setminus\varepsilon}^t)$ is randomly selected from a distribution $G_{I^t\setminus\varepsilon}(\rho)$.

Assumption 4. $G_{I^t\setminus\varepsilon}(\rho)$ has full support on $\Pi(I_{\setminus\varepsilon}^t)$.

When a set of players S chooses new strategies, we allow a small amount of public randomization to take place when it comes to selecting the strategies that will in fact be played. This is to circumvent an issue introduced by discretization and is important for Theorem 2 but unnecessary for Theorem 1. Let Q be the set of all distributions on $(d,P)_S$ such that:

$$\forall q \in Q\colon\quad (\tilde{d},\tilde{P})_S, (\bar{d},\bar{P})_S \in \operatorname{supp}(q) \quad\Longrightarrow\quad \forall i,\quad |\tilde{d}_i - \bar{d}_i| \in \{0,\delta\},\quad \tilde{P}_i = \bar{P}_i$$

If public randomization is disallowed and we restrict the support of every $q \in Q$ to be a singleton then Theorem 1 still holds. What randomization in effect allows is for a player to make demands for fractions of δ. Where possible, S will choose strategies so that each $i \in S$ obtains at least $\underline{u}_i^t$ and the strategies are Pareto efficient with respect to the payoffs of players in S. Define:

$$\begin{aligned}
\mathcal{D}_S^{1t} &= \big\{q \in Q\colon \forall i \in S,\ \mathbb{E}_q\big[U_i\big((d,P)_S|\sigma_i^t\big)\big] \geqslant \underline{u}_i^t\big\}\\
\mathcal{D}_S^{2t} &= \big\{q \in Q\colon \nexists \tilde{q} \in Q \text{ such that}\\
&\qquad \mathbb{E}_{\tilde{q}}\big[U_i\big((d,P)_S|\sigma_i^t\big)\big] \geqslant \mathbb{E}_q\big[U_i\big((d,P)_S|\sigma_i^t\big)\big]\ \forall i \in S,\\
&\qquad \mathbb{E}_{\tilde{q}}\big[U_i\big((d,P)_S|\sigma_i^t\big)\big] > \mathbb{E}_q\big[U_i\big((d,P)_S|\sigma_i^t\big)\big] \text{ for some } i \in S\big\}
\end{aligned}$$

When players in S choose from $\mathcal{D}_S^{1t} \cap \mathcal{D}_S^{2t}$ the outcome will be from a set $\mathcal{D}_S^t$.

$$\mathcal{D}_S^t = \big\{(d,P)_S\colon \exists q \in \mathcal{D}_S^{1t} \cap \mathcal{D}_S^{2t} \text{ with } (d,P)_S \in \operatorname{Supp}(q)\big\}$$

Assumption 5. For each $S \in \rho^t$, if $\mathcal{D}_S^t$ is nonempty, $(d^t,P^t)_S$ is chosen from a distribution with full support over $\mathcal{D}_S^t$.

$\mathcal{D}_S^t$ comprises the outcomes which can arise when S chooses Pareto efficient sets of strategies that give all $j \in S$ an expected payoff weakly higher than $\underline{u}_j^t$. When calculating their expected utility given $(d, P)_S$, a player in S takes expectations over his distribution on the actions of players outside S.

Assumption 6. If $\mathcal{D}_S^t = \emptyset$ all $i \in S$ best respond as they would in standard adaptive play, that is they play:

$$\left(d_i^t, P_i^t\right) \in \arg \max_{\substack{P_i \subseteq I \\ d_i \in \Sigma_i^\delta}} U_i\left((d_i, P_i)|\sigma_i^t\right)$$

By the description above it can be seen that the strategies chosen are binding for the current period only. No agreements are made which bind the actions of players in future periods. This is in keeping with the interpretation of the game as one being played by representative agents plucked from populations on a period by period basis: no agent can make a commitment as regards the future actions of agents playing in the same position in the game. This interpretation of the model also justifies the lack of foresight in the players' choices of strategy: it makes sense for them to act as myopic best responders if they are unlikely to be playing the game again in the near future.

3.3. Game stage

The game Γ described in Section 2 is then played using strategies (d^t, P^t) determined as above.

Define $\Omega = \mathcal{H} \times \mathcal{K}$, $\omega^t = (h^t, \varkappa^t) \in \Omega$. The perturbed process with $\varepsilon > 0$ gives an aperiodic Markov process $\mathcal{M}_\Omega^\varepsilon$ over Ω with transition kernel $\mathbf{P}_\varepsilon(x, A)$, $x \in \Omega$, $A \subseteq \Omega$. $\mathcal{M}_\Omega^\varepsilon$ has a unique ergodic distribution $\lambda_\Omega^\varepsilon$. The unperturbed process is denoted $\mathcal{M}_\Omega$ with transition probabilities $\mathbf{P}(x, A)$. Theorem 1 concerns the absorbing allocations of $\mathcal{M}_\Omega$. Theorem 2 concerns the stochastically stable states, the $\omega \in \Omega$ for which $\lim_{\varepsilon \to 0} \lambda_\Omega^\varepsilon(\omega) > 0$.

Note that the process restricted so that $|S| = 1\ \forall S \in \rho^t$, $\forall t$ is effectively the process of [5]. If we further restrict the process so that $P_i = I\ \forall i \in I$ then we have a similar model to [4]. Finally, if in addition to these restrictions we have that $v(S) = 0\ \forall S \neq I$ we return to the model of [10]. Restrictions relating the process in this paper to those of [1] and [2] are discussed in Section 6.

4. Convergence

Definition. A state $\omega \in \Omega$ is a *convention* if $\mathbf{P}(\omega, \omega) = 1$. Let $\mathbf{C} \subset \Omega$ denote the set of conventions.

We show that under our learning dynamic convergence to the interior core is guaranteed when the interior core is nonempty: joint strategic switching enables the equivalence between core allocations and conventions which was shown in [4] to hold for convex characteristic functions to be extended to all superadditive characteristic functions.

Theorem 1. *If Assumptions 1–6 hold*:

(i) $\omega = (h, \varkappa) \in \mathbf{C}$ *if and only if* $h = (d, I)^m$ *for some* $d \in \mathcal{C}^*(v)$, $\varkappa = \mathbf{0}$.
(ii) *For all* $\omega \in \Omega$, $\mathbf{P}^t(\omega, \mathbf{C}) \to 1$ *as* $t \to \infty$.

The intuition for the proof of convergence is as follows. Starting from any set of players S who will only form coalitions with players within S, subsets of S who can gain higher payoffs on their own will separate from S. They do this by adjusting their strategies so as to commit not to form coalitions with players outside of the subset in question. This continues until we are left with subsets of players S_l who will only form coalitions with players within S_l and for whom any strict subset $T \subset S_l$ has its characteristic function inequality satisfied. Starting within such an S_l, a subset of players who can guarantee themselves a higher payoff by going into a coalition with players outside S_l do so, before being forgotten and left on their own by the players with whom they went into coalition, leaving the status quo ante only with S_l now split into two. This process of separation can continue until we are left with a partition $\{S_1, \dots, S_\alpha\}$ and demands such that the only characteristic function inequalities left unsatisfied involve unions of elements of the partition. All characteristic functions involving strict subsets of players from any S_l are satisfied. No players outside of a given S_l are concerned with the allocation within S_l, so all elements of the partition can be joined together and the surplus from creating $\bigcup_{i=1}^{n} S_l$ shared so that all characteristic functions are satisfied.

4.1. Example (Agastya [3])

We examine an example of a non-convex characteristic function.[4] Let $I = \{1, 2, 3, 4\}$, $v(\{i\}) = 0$, $v(\{i, j\}) = v(\{i, j, k\}) = 7$, $v(I) = 16$ where $i \neq j \neq k \in I$, $m = 2k_i = 2\ \forall i$, $u(d) = d$. This characteristic function is non-convex as:

$$v\big(\{1, 2, 3\}\big) + v\big(\{3\}\big) = 7 < 14 = v\big(\{1, 3\}\big) + v\big(\{2, 3\}\big)$$

Let the distribution over feasible coalitions be such that all feasible coalition structures are equally likely to form. Then the allocation $(3, 3, 3, 7)$ is a steady state for the processes described by [3] and [5], with the strategies in the latter formulation also specifying that every player is willing to form coalitions with every other player. This allocation is not a core allocation, for example $d_1 + d_2 = 3 + 3 < v(\{1, 2\}) = 7$. To see that it is strict Nash first note that player 4 cannot improve his payoff by adjusting his strategy as he cannot possibly do better outside the grand coalition and within the grand coalition his payoffs are bounded by $v(I)$ less the demands of the other players:

$$v(I) - d_1 - d_2 - d_3 = 16 - 3 - 3 - 3 = 7$$

If one of the other players, say player 1, were to deviate and demand $3 < d_1 \leqslant 4$ then the only coalitions which would be feasible would be $\{1, 2\}, \{2, 3\}, \{1, 3\}$, each of which would have an equal chance of forming. Thus player 1's expected payoff would be $\frac{2}{3}d_1 \leqslant \frac{2}{3}4 < 3$ so he will not deviate. If he were to deviate and demand $d_1 > 4$ then he would have no chance of being part of a feasible coalition and would earn expected payoff of zero.[5]

In our model this allocation is unstable. The partition $\rho = \{\{1, 2\}, \{3\}, \{4\}\}$ could form, leading players 1 and 2 to play:

$$d_i = \frac{7}{2}, \qquad P_i = \{1, 2\}, \quad i = 1, 2$$

[4] A characteristic function is said to be convex if $v(S \cup T) + v(S \cap T) \geqslant v(S) + v(T)$ for all $S, T \subseteq I$.

[5] Under Rozen's formulation a deviation to $d_1 = 4$ and $P_1 = \{1, 2\}$ would lead to $\{1, 2\}, \{2, 3\}$ forming with equal probability to give expected payoff to player 1 of $\frac{1}{2}4 < 3$. A deviation to $d_1 = 4$ and $P_1 = \{1, 2, 3\}$ or $P_1 = I$ would lead to $\{1, 2\}, \{2, 3\}, \{1, 3\}$ forming with equal probability to give expected payoff to player 1 of $\frac{2}{3}4 < 3$.

which would give both players a higher payoff than 3. However, this would cause players 3 and 4 to obtain payoffs of 0, so in the next period there would be a possibility of all the players agreeing to play $P_i = I$, $d_i = 4$ $\forall i$, which is an interior core allocation and stable. Note that $P_i = I$, $d = (3, 4, 4, 5)$ which is a core allocation but not an interior core allocation is not stable under our dynamic, because players 1 and 2 could switch to playing $P_i = \{1, 2\}$, thus breaking up the grand coalition.

4.2. *Example (Theorem 1)*

This example demonstrates the process which gives rise to Theorem 1. Let $I = \{1, 2, 3, 4\}$, $v(\{i\}) = 0$, $v(\{1, 2\}) = 3$, $v(\{3, 4\}) = 7$, $v(\{1, 3\}) = v(\{2, 3\}) = v(\{1, 4\}) = v(\{2, 4\}) = 0$, $v(\{1, 2, 3\}) = v(\{1, 2, 4\}) = 3$, $v(\{1, 3, 4\}) = v(\{2, 3, 4\}) = 11$, $v(I) = 16$ where $i \in I$, $m = 2k_i = 2$ $\forall i$. Say we start at time τ with $P_i^\tau = \{I\}$ for all players and $d_1^\tau = 5, d_2^\tau = 9, d_3^\tau = d_4^\tau = 1$. This is not in the interior core and so can be broken the next period as all players sample $\sigma_i^{\tau+1} = \{\tau\}$ and players 3 and 4 agree to play $P_i^{\tau+1} = \{3, 4\}, d_3^{\tau+1} = 3, d_4^{\tau+1} = 4$. Players 1 and 2 continue to play the same strategies as before and consequently earn zero payoff. The following period players 1 and 2 sample $\sigma_i^{\tau+2} = \{\tau + 1\}$ and agree to play $P_i^{\tau+2} = \{1, 2\}, d_1^{\tau+2} = 1$, $d_2^{\tau+2} = 2$. Players 3 and 4 sample $\sigma_i^{\tau+2} = \{\tau\}$ and play the same strategies as in period $\tau + 1$. Note that the characteristic function inequality for $\{1, 3, 4\}$ is not satisfied:

$$d_1^{\tau+2} + d_3^{\tau+2} + d_4^{\tau+2} = 1 + 3 + 4 = 8 < 11 = v\big(\{1, 3, 4\}\big)$$

Subsequently all players sample $\sigma_i^{\tau+3} = \{\tau + 2\}$ and players 1, 3 and 4 agree to play $P_i^{\tau+3} = \{1, 3, 4\}, d_1^{\tau+3} = 3, d_3^{\tau+3} = 4, d_4^{\tau+3} = 4$. The following period player 2 samples $\sigma_2^{\tau+4} = \{\tau + 3\}$ and plays $P_2^{\tau+4} = \{2\}$. Players 3 and 4 sample $\sigma_i^{\tau+4} = \{\tau + 2\}$ and return to playing $P_i^{\tau+4} = \{3, 4\}$, so that the coalition structure $\{\{1\}, \{2\}, \{3, 4\}\}$ forms, leaving players 1 and 2 once again earning zero payoff. Note that the only characteristic functions left unsatisfied are now those which involve unions of elements of the coalition structure. This means that the following period all players can sample $\sigma_i^{\tau+5} = \{\tau + 4\}$ and agree to $P_i^{\tau+5} = \{I\}$ $\forall i$ and an allocation in the interior core, for example $d_1^{\tau+5} = d_2^{\tau+5} = d_3^{\tau+5} = d_4^{\tau+5} = 4$.

5. Stochastic stability

From hereon in it is assumed that $\varepsilon > 0$. This section analyzes what happens as ε becomes small and determines the stochastically stable states of our model. The following technical assumption, under which Theorem 1 still holds, replaces Assumption 4 and is introduced to get rid of feedback from a player's own errors to his actions:

Assumption 7. $G_{I^t\backslash\varepsilon}(\rho)$ has full support on $\{\rho \in \Pi(I^t_{\backslash\varepsilon})\colon\ i, j \in S \in \rho,\ \tau \in \sigma_i^t,\ \varkappa_j^\tau = 1 \Longrightarrow |S| = 1\}$.

This assumption eliminates the possibility of groups of players responding to their own errors. Switches between conventions will be driven by groups of players responding to random errors made by players outside the group in question. An equally useful assumption for our purposes would be that if a player sampled a period in which he or one of his negotiating partners had made an error then he would discard that observation and draw another. Without this assumption it is always possible to switch between conventions with a single error.[6]

Definitions. $\omega \in \Omega$ is a *stochastically stable state* if and only if:

$$\lim_{\varepsilon \to 0} \lambda^{\varepsilon}_{\Omega}(\omega) > 0$$

Let $\alpha_i = \frac{k_i}{m}$ and fix α_i $\forall i$. An allocation d is a *stochastically stable allocation* if there exists a stochastically stable state with allocation d for arbitrarily large m.

α_i, a player's sample size as a proportion of m, can be thought of as a measure of his smartness. Utilities and characteristic functions are normalized so that $u(0) = 0$ and $v(\{i\}) = 0$ and assumed to satisfy:

Assumption 8. $u(.)$ is continuously differentiable.

Definition.

$$D^* = \left\{ d\colon\ d \in \arg \min_{d \in \mathcal{C}(v)} \max_{j \in I} \sum_{i \neq j} \frac{1}{\alpha_i} \left(\frac{\partial \log u_i(d_i)}{\partial d_i} \right)^{-1} \right\}$$

Recall the definition of δ as the smallest money unit which was given in Section 2. The following theorem characterizes stochastically stable allocations as δ becomes small and gives a condition under which there is a unique limit of stochastically stable allocations.

Theorem 2. *Let $\{d^n\}_{\delta_n > 0}$ be any convergent infinite sequence of stochastically stable allocations with $\delta_n \to 0$. Under Assumptions 1–3, 5–8:*

(i) $\lim_{\delta_n \to 0} d^n \in D^*$.
(ii) *If $\frac{u_i(.)}{u_i'(.)}$ is strictly convex for all $i \in I$ then D^* is a singleton.*

Note that D^* is defined for the core rather than for the interior core. This is because limits of sequences of interior core allocations will not necessarily be in the interior core, but will be in the core. The proof shows that, from any convention, as $\delta \to 0$, the easiest way to leave the convention is for a single player to make errors, demanding a small amount more than he receives in the current convention, following which the other players drop their demands (in expectation) by a small amount to ensure that their demands continue to be met. By jointly reducing their demands they share the utility loss of receiving less than they did before. It is then proven that once this process has started it is possible to transition to any other convention. Minimum cost spanning tree methods of [7] can then be used to show that limits of stochastically stable states must solve the optimization problem in the definition of D^*.

[6] Without Assumption 7 it could happen that from a convention with $h^t = (d, I)^m$, $d \in \mathcal{C}^*(v)$, a player $j \in I$ makes an error, demanding $d_j + \delta$. If in the next period player i samples player j's error and $S = \{i, j\} \in \rho^{t+1}$ then both players could achieve expected payoffs greater than $\underline{u}_i^{t+1}, \underline{u}_j^{t+1}$ respectively if with small probability a they play $d_i^{t+1} = d_i - \delta$, $d_j^{t+1} = d_j + \delta$ and with probability $1 - a$ they play $d_S^{t+1} = d$. Movement between adjacent conventions would require only a single error and all conventions would be stochastically stable.

When marginal utility decreases with wealth, players who do better in a convention are less reluctant to reduce their demands. This suggests that the easier ways to leave a convention involve richer players responding to the errors of poorer players and that more egalitarian conventions may be more stable.[7] In fact, for a cooperative game where the amount of good received is identical to the agents' utilities or the utilities equal the allocation of good raised to some positive power (including CRRA utility) the following result follows directly from Theorem 2.

Corollary. *If $u_i(d) = ad^b$, $a, b \in \mathbb{R}_{++}$, $\alpha_i = \alpha$ for all $i \in I$ then limits of stochastically stable allocations maximize the Rawlsian social welfare function*:

$$d \in \arg \max_{d \in \mathcal{C}(v)} \min_{i \in I} d_i$$

So the dynamic process and stochastic stability characterization of this paper provide an underpinning to the Rawlsian social welfare function. If α_i differ between agents then the allocations in the objective function are weighted by the reciprocal of α_i so that the amount of good agent i receives is weakly increasing in α_i: it pays to be smart. Theorem 2 is now further examined through some examples.

5.1. Example

Let $I = \{1, 2, 3, 4\}$, $v(\{1\}) = v(\{2\}) = 0$, $v(\{3\}) = 20$, $v(\{4\}) = 40$, $v(\{1, 2\}) = 0$, $v(\{2, 3\}) = 70$, $v(\{3, 4\}) = 100$, $v(\{1, 3\}) = 20$, $v(\{1, 4\}) = v(\{2, 4\}) = 40$, $v(\{1, 2, 3\}) = 70$, $v(\{1, 2, 4\}) = 40$, $v(\{1, 3, 4\}) = 100$, $v(\{2, 3, 4\}) = 110$, $v(I) = 120$.

Take $u_i(d_i) = \log(1 + d_i)$, $\alpha_i = \alpha\ \forall i$. Then the allocation $(10, 10, 60, 40) \in D^*$. Moreover, $u_i(d_i)/u_i'(d_i) = (1 + d_i)\log(1 + d_i)$ is strictly convex so the allocation is in fact the limit of any sequence of stochastically stable allocations as $\delta \to 0$. As δ gets small, all stochastically stable allocations are close to $(10, 10, 60, 40)$.

Now take $u_i(d_i) = 1 - e^{-d_i}$, $\alpha_i = \alpha\ \forall i$. We again have strict convexity of $u_i(d_i)/u_i'(d_i) = 2(e^{d_i} - 1)$, however now the unique limit is the allocation $(0, 20, 50, 50)$. To raise the minimum value of $(\alpha_j \partial \log u_j(d_j)/\partial d_j)^{-1}$, $j \in \{1, 2, 3, 4\}$, requires that player 3 be paid above 50 and the exponential term in the objective function ensures that this is too expensive.

Concave $u(.)$ implies increasing $u_i(d_i)/u_i'(d_i)$ which implies that if the allocation of the poorest player can be increased without increasing the allocations of any of the other players then doing so makes an allocation more stable. Also, strictly convex $u_i(d_i)/u_i'(d_i)$ implies that if we keep the allocation of the poorest player constant then stability is increased by lowering wealth inequality amongst the other $N - 1$ players.

5.2. Example

Let $I = \{1, 2, 3\}$, $v(\{1\}) = 60$, $v(\{2\}) = v(\{3\}) = 0$, $v(\{1, 2\}) = 100$, $v(\{1, 3\}) = 60$, $v(\{2, 3\}) = 0$, $v(\{1, 2, 3\}) = 120$.

[7] A similar result in [4] characterizes stable allocations as minimizing the maximum weighted payoff amongst players. Joint strategic switching allows the payoffs of *all* players to determine the stable allocation. [10] predicts the Nash bargaining solution. The results of [14] which predict the Kalai–Smorodinsky bargaining solution are for a different model where if players fail to coordinate on a specific division of surplus then they obtain nothing.

If $\alpha_i = \alpha\ \forall i$, then concave utility which gives strictly convex $u_i(d_i)/u_i'(d_i)$ will select the limit allocation $(60, 40, 20)$. However, if $u_i(d_i) = d_i^{1-\upsilon}/(1-\upsilon)$, $\upsilon > 0$, then the corollary to Theorem 2 applies and $d \in D^*$ maximize the minimum allocation, player 3's in this case, subject to fulfilling the constraints of the characteristic function. So all allocations $(60 + x, 40 - x, 20)$, $x \in [0, 20]$ are in D^*. However, only the allocation $(80, 20, 20)$ is a limit of stochastically stable states. This is because $(1/\delta[u_i(d_i) - u_i(d_i - \delta)/u_i(d_i)])^{-1}$ is concave and only becomes linear in the limit.[8] However, if players have CRRA utility and some existing wealth ξ_i, then their normalized utility $u_i(d_i) = ((d_i + \xi_i)^{1-\upsilon} - \xi_i^{1-\upsilon})/(1-\upsilon)$ gives convex $u_i(d_i)/u_i'(d_i)$.

6. Discussion

6.1. Discretization and randomization

The small amount of randomization in the definition of $\mathcal{D}_S^t$ merits some comment. The model involves players sometimes sharing gains and losses. Sharing the smallest possible amount is by definition impossible. However, given that we are interested in the limit as the smallest possible amount approaches zero, we would like it to be possible. This problem is circumvented by introducing the possibility of randomization, which can be taken literally or as an approximation for what happens when dealing with quantities smaller than δ. It is worth noting that as utility is continuously differentiable and vNM, the utility of the randomization $a_i u_i(d_i - \delta) + (1 - a_i)u_i(d_i)$ approaches $u_i(d_i - a_i\delta)$ as $\delta \to 0$, implying that the randomization is reasonable in the context in which it is used in this paper.

6.2. Relation to Green [1] and Feldman [2]

The models of [1] and [2] have $m = k_i = 1$ and the restriction that each period, for some $S \in \rho^t$, if $\mathcal{D}_S^t \neq \emptyset$ then $T \in \rho^t \implies P_i = T\ \forall i \in T$: players who discuss strategy must form coalitions with one another. If $\mathcal{D}_S^t = \emptyset$ then $(d, P)^t = (d, P)^{t-1}$. [1] also imposes $\rho^t = \{S, I \setminus S\}$, $i \in I \setminus S \implies \underline{u}_i^t < v(\{i\})$ (implied by Assumptions 7 and 1 of [1] respectively). [2] imposes the alternative restriction that $\rho^t = \{S, \{i\}, \ldots, \{j\}\}$. Both processes are asymmetric: some set S acts in a similar fashion to the process in the current paper, whereas players outside S are treated differently. In [1], players in $I \setminus S$ will accept any Pareto strategies and effectively have no reservation utility. In [2], players in $I \setminus S$ are assumed to remain singletons.

6.3. Further direction

A natural extension of this paper would be for the methodology to be applied to cases where the underlying cooperative game is in partition function form and the set of attainable payoffs for any coalition S that forms in a given period depends on the coalition structure of players outside of S. The payoffs of groups of players in the current model already depend on the strategies of other players as these strategies help to determine which coalition structures will form in each period. However, a superadditivity assumption makes less intuitive sense when payoffs depend on partitions and results of interest may be obtained when this is weakened.

[8] If it were the case that $\upsilon < 0$ then this expression would be convex and become linear in the limit, again selecting $(60, 40, 20)$ as the only limit of stable allocations.

Appendix A

Proof of Theorem 1. First the *if* part of (i) is proven.

$$\omega^{t-1} = \omega = \big((d, I)^m, \mathbf{0}\big) \quad \text{for some } d \in \mathcal{C}^*(v) \quad \Longrightarrow \quad \underline{u}_i^t = u_i(d_i) \quad \forall i \in I$$
$$\forall S \in \rho^t, \quad U_i\big((d^t, P^t)_S | \sigma_i^t\big) \geqslant \underline{u}_i^t \quad \forall i \in S \quad \Longrightarrow \quad U_i\big((d^t, P^t)_S, (d, I)_{I \setminus S}\big) \geqslant u_i(d_i)$$
$$\Longrightarrow \quad (d^t, P^t)_S = (d, I)_S \quad \Longrightarrow \quad \mathbf{P}(\omega, \omega) = 1 \quad \Longrightarrow \quad \omega \in \mathbf{C}$$

The *only if* part of (i) is proven if the convergence of part (ii) of the theorem takes the process to states of the requisite form. Assume $k_i = 1$ for all i, $m \geqslant 2$. To expand the proof to all k_i simply allow each strategy profile in the proof to be repeated $\bar{k} = \max_i k_i$ times. Let $\underline{d}_i^t := u^{-1}(\underline{u}_i^t)$.

Period 1 Let (d^1, P^1): $d^1 \notin \mathcal{C}^*(v)$ or $P^1 \neq I^N$. With positive probability the following events occur.

Period 2 Let $\varrho = \{S_1, S_2, \ldots, S_\alpha\}$ be such that: $i \in S_l \Longrightarrow P_i^1 \subseteq S_l$. $\forall i \in I$: $\sigma_i^2 = \{1\}$. Let:

$$\mathcal{S}_1 := \Big\{ S \subset I \text{ s.t. } \exists S_n \in \varrho\colon\ S \subset S_n,\ v(S) \geqslant \sum_{i \in S} \underline{d}_i^2 \Big\}$$
$$\mathcal{S}_2 := \Big\{ S \subset I\colon\ \exists S_n \in \varrho \text{ s.t. } S_n \cap S \neq \emptyset,\ S_n \nsubseteq S,\ v(S) \geqslant \sum_{i \in S} \underline{d}_i^2 \Big\}$$
$$\mathcal{S} := \begin{cases} \mathcal{S}_1 & \text{if } \mathcal{S}_1 \neq \emptyset \\ \mathcal{S}_2 & \text{if } \mathcal{S}_1 = \emptyset,\ \mathcal{S}_2 \neq \emptyset \\ \emptyset & \text{if } \mathcal{S}_1 = \emptyset,\ \mathcal{S}_2 = \emptyset \end{cases}$$
$$T \in \arg\max_{S \in \mathcal{S}} v(S) - \sum_{i \in S} \underline{d}_i^2$$

$\rho^2 = \{T, S_1 \setminus T, S_2 \setminus T, \ldots, S_\alpha \setminus T\}$. For some $j \in S_n \cap T$: $P_j^2 = T$, $d_j^2 = d_j^1 + v(T) - \sum_{i \in T} \lceil \underline{d}_i^2/\delta \rceil \delta$. $\forall i \in T, i \neq j$: $P_i^2 = T$, $d_i^2 = \lceil \underline{d}_i^2/\delta \rceil \delta$. Then $(d^2, P^2)_T \in \mathcal{D}_T^2$. $\forall l, \forall i \in S_l \setminus T$, $P_i^2 \subseteq S_l$.

Period 3 $\rho^3 = \{S_1, S_2, \ldots, S_n \setminus T, S_n \cap T, \ldots, S_\alpha\}$. $\forall i \in S_n \cap T$: $\sigma_i^3 = \{2\}$, $P_i^3 \subseteq S_n \cup T$. $\forall i \in S_n \setminus T$: $\sigma_i^3 = \{2\}$, $P_i^3 \subseteq S_n \setminus T$. $\forall l, \forall i \in S_l, l \neq n$: $\sigma_i^3 = \{1\}$, $P_i^3 \subseteq S_l$.

Period 4 $\rho^4 = \{S_1, S_2, \ldots, S_n \setminus T, S_n \cap T, \ldots, S_\alpha\}$. $\forall i \in S_n \cap T$: $\sigma_i^4 = \{3\}$, $P_i^4 \subseteq S_n \cap T$. $\forall i \in S_n \setminus T$: $\sigma_i^4 = \{2\}$, $P_i^4 \subseteq S_n \setminus T$. $\forall i \in S_l, l \neq n$: $\sigma_i^4 = \{3\}$, $P_i^4 \subseteq S_l \cup T$.

Period 5 $\rho^5 = \{S_1, S_2, \ldots, S_n \setminus T, S_n \cap T, \ldots, S_\alpha\}$. $\forall i \in S_n \cap T$: $\sigma_i^5 = \{3\}$, $P_i^5 \subseteq S_n \cap T$. $\forall i \in S_n \setminus T$: $\sigma_i^5 = \{4\}$, $P_i^5 \subseteq S_n \setminus T$. $\forall i \in S_l, l \neq n$: $\sigma_i^5 = \{4\}$, $P_i^5 \subseteq S_l$.

Period 6 to τ Reindexing $\{S_1, S_2, \ldots, S_n \setminus T, S_n \cap T, \ldots, S_\alpha\}$ as $\{S_1, S_2, \ldots, S_\alpha\} =: \varrho$, periods 2 to 5 can be repeated until $\mathcal{S} = \emptyset$. This must happen as ϱ becomes a finer partition with each iteration and if ϱ were to be composed entirely of singletons then $\mathcal{S} = \emptyset$.

Period $\tau + 1$ $\forall i$: $\sigma_i^{\tau+1} = \{\tau\}$ so that $\mathcal{S} = \emptyset$ and

$$v(S) \geqslant \sum_{i \in S} \underline{d}_i^{\tau+1} \quad \Longrightarrow \quad S = \bigcup_{l \in \mathcal{L}} S_l \quad \text{for some} \quad \mathcal{L} \subseteq \{1, \ldots, \alpha\}$$

$\rho^{\tau+1} = \{I\}$. $\forall i$: $P_i^{\tau+1} = I$, $d_i^{\tau+1} \geqslant \underline{d}_i^{\tau+1}$. $d^{\tau+1} \in \mathcal{C}^*(v)$. To see that such an allocation exists first note that for $d \in \mathcal{C}^*(v)$, for any S_l we must have $\sum_{i \in S_l} d_i > v(S_l) \geqslant \sum_{i \in S_l} \underline{d}_i^{\tau+1}$. For

$S = \bigcup_{l \in \mathcal{L}} S_l$ it must be that $\sum_{i \in S} d_i > v(S) \geqslant \sum_{S_l \subseteq S} v(S_l) \geqslant \sum_{S_l \subseteq S} \sum_{i \in S_l} \underline{d}_i^{\tau+1}$. If this is satisfied it remains so when we change d_i, $i \in S_l$, while leaving $\sum_{i \in S_l} d_i$ unchanged. So we choose $d_i \geqslant \underline{d}_i^{\tau+1}$ $\forall i$ which implies that $\sum_{i \in S} d_i > v(S)$ for all $S \neq \bigcup_{l \in \mathcal{L}} S_l$. So $\mathcal{C}^*(v) \neq \emptyset \Longrightarrow \{d \in \mathcal{C}^*(v)\colon d_i \geqslant \underline{d}_i^{\tau+1}\ \forall i\} \neq \emptyset$. $\square$

To prove Theorem 2 we begin by showing that for any given interior core allocation, for small enough δ the least cost (in terms of random errors) way of inducing a best response from any set of players which differs from a current convention is via $N-1$ player best responses to individual errors. That is to say: one player mutates until every other player agrees together to best respond in a way that differs from their conventional strategies.

Lemma 1. *Let $j \in \arg\max_{i \in I} k_i[\frac{u_i(d_i) - u_i(d_i - \delta)}{u_i(d_i)}]$. Let $d \in \mathcal{C}^*(v)$. Define $\mu^*(d), a_i > 0$, $i \neq j$, as the unique constants which solve $\sum_{i \neq j} a_i = 1$ and*:

$$k_i a_i \left[\frac{u_i(d_i) - u_i(d_i - \delta)}{u_i(d_i)}\right] = \mu^*(d) \quad \forall i \in I,\ i \neq j$$

Then $\exists \delta^* > 0$ such that $\forall \delta \in (0, \delta^*)$, from convention $(d, I)^m$, the minimum number of random errors required for any set of players to respond in a way that differs from conventional demand d is equal to $\lceil \mu^*(d) \rceil$. $\mu^*(d)$ can also be expressed as:

$$\mu^*(d) = \min_{j \in I} \left(\sum_{i \neq j} \frac{1}{k_i}\left[\frac{u_i(d_i)}{u_i(d_i) - u_i(d_i - \delta)}\right]\right)^{-1}$$

$$= \min_{j \in I} \left(\sum_{i \neq j} \frac{1}{\alpha_i}\left[\frac{u_i(d_i)}{u_i(d_i) - u_i(d_i - \delta)}\right]\right)^{-1} m$$

Proof. Let $h^{\tau'} = (d, I)^m$ be the convention at time τ'. Let $\tilde{\omega}$ be the first state, say at time $\tau > \tau'$ at which a set of players S is the first set of players to play actions which differ from $(d, I)_S$ without making any random errors. Let $(d, P)_S^*$ denote these actions. $(d, P)_S^* \in \operatorname{supp}(q)$ for some $q \in \mathcal{D}_S^{1\tau} \cap \mathcal{D}_S^{2\tau}$. For each $i \in S$ there must exist a sample σ_i^τ, $|\sigma_i^\tau| = k_i$ such that:

$$\mathbb{E}_q\left[U_i\left((d, P)_S^* | \sigma_i^\tau\right)\right] \geqslant \underline{u}_i^\tau = \frac{1}{k_i} \sum_{j \in \sigma_i^\tau} U_i\left(\left(d^j, P^j\right)\right) = U_i\left((d, I)_S | \sigma_i^\tau\right) \quad \forall i \in S,\ |S| \geqslant 2$$

where the final equality follows as Assumption 7 $\Longrightarrow (d^j, P^j)_i = (d, I)_i$ $\forall i \in S,\ j \in \sigma_i^\tau$. For $|S| = 1$, Pareto efficiency implies that $\mathbb{E}_q[U_i((d, P)_S^* | \sigma_i^\tau)] \geqslant U_i((d, I)_S | \sigma_i^\tau)$. Let there be L_i observations in σ_i^τ where the actions of at least one player differ from those of (d, I), $L_i := |\{t \in \sigma_i^\tau\colon (d^t, P^t) \neq (d, I)\}|$ Without loss of generality assume these are the most recent observations from σ_i: $\{t \in \sigma_i^\tau\colon (d^t, P^t) \neq (d, I)\} = \{\tau - L_i, \ldots, \tau - 1\}$. Note that by the definition of $\tilde{\omega}$, each of the sampled observations not equal to (d, I) involves at least one error.

Case 1. ($\sum_{i \in S} d_i^* < \sum_{i \in S} d_i$). We have:

$$\mathbb{E}_q\left[u_i\left(d_i^*\right)\right] \geqslant \mathbb{E}_q\left[U_i\left((d, P)_S^* | \sigma_i^\tau\right)\right] \geqslant U_i\left((d, I)_S | \sigma_i^\tau\right) \geqslant \left(1 - \frac{L_i}{k_i}\right) u_i(d_i)$$

Where the first inequality is because a player can never do better than to obtain his demand with certainty, the second inequality follows by hypothesis and the third from player i's subjective probability of $(d, I)_{I\setminus S}$ being played. Rearranging we get:

$$L_i \geqslant k_i\left[1 - \frac{\mathbb{E}_q[u_i(d_i^*)]}{u_i(d_i)}\right]$$

which holds for all $i \in S$, giving $\max_{i\in S}\lceil k_i[1 - \frac{\mathbb{E}_q[u_i(d_i^*)]}{u_i(d_i)}]\rceil$ as a lower bound on the number of errors in h^τ. This bound decreases in $\mathbb{E}_q[u_i(d_i^*)]$ $\forall i \in S$ and is lowest when $\sum_{i\in S} d_i^* = \sum_{i\in S} d_i - \delta$, $\mathbb{E}_q[u_i(d_i^*)] < u_i(d_i)$ $\forall i \in S$. This implies distributions q where one member of S decreases his demand by δ and each $i \in S$ is this one member with some probability a_i. So:

$$L_i \geqslant k_i\left[1 - \frac{(1-a_i)u_i(d_i) + a_i u_i(d_i - \delta)}{u_i(d_i)}\right] = k_i a_i\left[\frac{u_i(d_i) - u_i(d_i-\delta)}{u_i(d_i)}\right] =: \mu_i(a_i, d)$$

To find an absolute lower bound this must be minimized over all sets S, that is:

$$\min_{\substack{S\in\mathcal{I}\\ a_i\in(0,1]}} \max_{i\in S} \mu_i(a_i, d) \quad \text{s.t.} \quad \sum_{i\in S} a_i = 1$$

This is minimized for a_i chosen so that $\mu_i(a_i, d)$ are the same for all $i \in S$: $\mu_i(a_i, d) =: \mu(d)$, S is as large as possible: $|S| = N - 1$, and the player j excluded from S satisfies $j \in \arg\max_{i\in I} k_i[\frac{u_i(d_i)-u_i(d_i-\delta)}{u_i(d_i)}]$.

As:

$$k_i a_i\left[\frac{u_i(d_i) - u_i(d_i-\delta)}{u_i(d_i)}\right] = \mu(d) \quad \forall i \in I,\ i \neq j$$

We have:

$$1 = \sum_{i\neq j} a_i = \sum_{i\neq j} \mu(d)\frac{1}{k_i}\left[\frac{u_i(d_i)}{u_i(d_i) - u_i(d_i-\delta)}\right]$$

which gives:

$$\mu(d) = \left(\sum_{i\neq j}\frac{1}{k_i}\left[\frac{u_i(d_i)}{u_i(d_i) - u_i(d_i-\delta)}\right]\right)^{-1}$$

and:

$$\mu^*(d) = \min_{j\in I}\left(\sum_{i\neq j}\frac{1}{k_i}\left[\frac{u_i(d_i)}{u_i(d_i) - u_i(d_i-\delta)}\right]\right)^{-1}$$

Case 2. $(\sum_{i\in S} d_i^* \geqslant \sum_{i\in S} d_i)$.

If $\exists i \in S$ with $\mathbb{E}[d_i^*] < d_i$ then for there to be a lower bound lower than in Case 1 it is necessary that $\mathbb{E}_q[d_i^*] > d_i - \delta$. So

$$\begin{aligned} &\mathbb{E}_q\big[u_i\big(d^*\big)\big] \geqslant \underline{u}_i^\tau \\ \implies\quad & a u_i(d_i - \delta) + (1-a)\left(1 - \frac{L_i}{k_i}\right)u_i(d_i) \geqslant \left(1 - \frac{L_i}{k_i}\right)u_i(d_i) \quad \text{for some } a \in (0,1) \\ \implies\quad & L_i \geqslant k_i\left(\frac{u_i(d_i) - u_i(d_i-\delta)}{u_i(d_i)}\right) \end{aligned}$$

which is still not as low as the bound in Case 1. Now assume $d^*_S \geqslant d_S$. Then $\forall i \in S$:

$$\frac{L_i}{k_i}u_i\big(v(I)\big) \geqslant \mathbb{E}_q\big[U_i\big((d,P)^*_S|\sigma_i^\tau\big)\big] \geqslant U_i\big((d,I)_S|\sigma_i^\tau\big) \geqslant \left(1-\frac{L_i}{k_i}\right)u_i(d_i)$$

Rearranging we get:

$$L_i \geqslant \frac{k_i u_i(d_i)}{u_i(v(I))+u_i(d_i)}$$

For small enough δ this is greater than the lower bound from Case 1. □

Note that coalitional behavior does not decrease the lower bound in Case 2 in the proof of Lemma 1. The reason is that if one player increases his demand he risks breaking the grand coalition and thus imposes a negative externality on the other players. In Case 1 above it was the case that if one player decreased his demand he imposed a positive externality on the other players (reduced the risk of breaking the grand coalition) who were therefore willing to share (probabilistically) the costs of taking such action.

The next step is to show that there exists a path from every convention $(d,I)^m$ to any other convention which requires exactly $\mu^*(d)$ errors.

Lemma 2. *Let $d^\dagger, d^\ddagger \in \mathcal{C}^*(v)$. Then the transition $(d^\dagger, I)^m \to (d^\ddagger, I)^m$ requires at most $\lceil \mu^*(d^\dagger)\rceil$ random errors.*

Proof.

Period $\tau - 2\bar{k} + 1$ to τ: $\forall i \in I$: $(d^t, P^t)_i = (d_i^\dagger, I)$. With positive probability:

Period $\tau + 1$ to $\tau + \lceil\mu^*(d^\dagger)\rceil$: Player $e \in \arg\max_{i\in I} k_i[\frac{u_i(d_i)-u_i(d_i-\delta)}{u_i(d_i)}]$ makes errors and plays $(d^t, P^t)_e = (d_e^\dagger + \delta, I)$. $\forall i \in I \setminus \{e\}$: $\sigma_i^t \subseteq \{\tau - \bar{k} + 1, \dots, \tau\}$, $(d^t, P^t)_i = (d_i^\dagger, I)$.

Period $\tau + \lceil\mu^*(d^\dagger)\rceil + 1$ to $\tau + \bar{k}$: $\forall i \in I$: $\sigma_i^t \subseteq \{\tau - \bar{k} + 1, \dots, \tau\}$, $(d^t, P^t)_i = (d_i^\dagger, I)$.

Period $\tau + \bar{k} + 1$ to $\tau + 2\bar{k}$: $\rho^t = \{I \setminus \{e\}, \{e\}\}$. $\sigma_e^t \subseteq \{\tau + 1, \dots, \tau + \bar{k}\}$, $(d^t, P^t)_e = (d_e^\dagger, I)$. $I \setminus e$ respond as in Lemma 1. Assume that j always turns out to be the player who reduces his demand by δ. $\forall i \in I \setminus \{e\}$: $\sigma_i^t \supseteq \{\tau + 1, \dots, \tau + \lceil\mu^*(d^\dagger)\rceil\}$. For some $j \in I \setminus \{e\}$: $(d^t, P^t)_j = (d_j^\dagger - \delta, I)$. $\forall i \in I \setminus \{e, j\}$: $(d^t, P^t)_i = (d_i^\dagger, I)$.

Period $\tau + 2\bar{k} + 1$ to $\tau + 3\bar{k}$: $\rho^t = \{\{1\}, \dots, \{N\}\}$. $\forall i \in I$: $\sigma_i^t \subseteq \{\tau + \bar{k} + 1, \dots, \tau + 2\bar{k}\}$. $(d^t, P^t)_j = (d_j^\dagger, I)$. $\forall i \in I \setminus \{j\}$: $(d^t, P^t)_i = (d_i^\dagger + \delta, I)$. Note that $d^t \geqslant d^\dagger$, $d^\dagger \in \mathcal{C}^*(v) \Longrightarrow U_i((d^t, P^t)) = 0\ \forall i$.

Period $\tau + 3\bar{k} + 1$ to $\tau + 4\bar{k}$: $\rho^t = \{I\}$. $\forall i \in I$: $\sigma_i^t \subseteq \{\tau + 2\bar{k} + 1, \dots, \tau + 3\bar{k}\}$. $(d^t, P^t)_i = (d_i^\ddagger, I)$.

Period $\tau + 4\bar{k} + 1$ to $\tau + 3\bar{k} + m$: $\forall i \in I$: $\sigma_i^t \subseteq \{t - \bar{k}, \dots, t - 1\}$. $(d^t, P^t)_i = (d_i^\ddagger, I)$. So $h^{\tau+3\bar{k}+m} = (d_i^\ddagger, I)^m$. □

For a convention $\hat{\omega}$ with demands $\hat{d}$, define $\beta(\hat{d})$ as the lowest number of random errors necessary to move the process to any other convention $\omega \neq \hat{\omega}$. Lemmas 1 and 2 have shown that for any $\hat{d}$ there exists a $\hat{\delta}$ such that for all $\delta < \hat{\delta}$: $\beta(\hat{d}) = \mu^*(\hat{d})$. We now show that for any allocation d which does not maximize $\mu^*(.)$ we can find δ small enough so that d is not stable.

Lemma 3. *Let $d^\dagger, d^\ddagger \in \mathcal{C}^*(v)$. If $\beta(d^\dagger) = \mu^*(d^\dagger)$, then $\mu^*(d^\dagger) > \mu^*(d^\ddagger)$ implies that $d^\ddagger$ is not a stochastically stable allocation.*

Proof. Note that for large m, $\mu^*(d^\dagger) > \mu^*(d^\ddagger) \Longrightarrow \lceil \mu^*(d^\dagger) \rceil > \lceil \mu^*(d^\ddagger) \rceil$. Define a graph on $\mathcal{C}^*(v)$. Directed edges between any two vertices of the graph are given a cost equal to the minimum number of random errors required to move from one vertex (state) to the other in the perturbed dynamic. [7] shows that any stochastically stable state is the root of a spanning tree of the graph which has minimum cost, where the cost of the tree is given by the sum of the costs of its edges. Take any spanning tree rooted at $d^\ddagger$. Replace the edge of the tree leaving $d^\dagger$ with an edge from $d^\dagger$ to $d^\ddagger$. By Lemma 2 the new tree will have the same cost. Reversing the direction of the edge between $d^\dagger$ and $d^\ddagger$ we obtain a tree rooted at $d^\dagger$ with lower cost.

$$Cost(d^\dagger) \leqslant Cost(d^\ddagger) - \lceil \mu^*(d^\dagger) \rceil + \lceil \mu^*(d^\ddagger) \rceil < Cost(d^\ddagger)$$

so $d^\ddagger$ cannot be stochastically stable. □

We now show what happens to the results of the previous lemmas when δ gets very small. Stochastically stable allocations are characterized as minima of a function and uniqueness is shown under a condition on the utility functions. This completes the proof of Theorem 2.

Proof of Theorem 2. Define:

$$\xi(d, \delta) := \frac{\mu^*(d)}{m\delta}$$

As:

$$\lim_{\delta \to 0} \frac{\alpha_i}{\delta} \left[\frac{u_i(d_i) - u_i(d_i - \delta)}{u_i(d_i)} \right] = \alpha_i \frac{\partial \log u_i(d_i)}{\partial d_i}$$

We have:

$$\begin{aligned} \lim_{\delta \to 0} \xi(d, \delta) = \lim_{\delta \to 0} \frac{\mu^*(d)}{m\delta} &= \lim_{\delta \to 0} \min_{j \in I} \left(\sum_{i \neq j} \frac{\delta}{\alpha_i} \left[\frac{u_i(d_i)}{u_i(d_i) - u_i(d_i - \delta)} \right] \right)^{-1} \\ &= \min_{j \in I} \left(\sum_{i \neq j} \left(\alpha_i \frac{\partial \log u_i(d_i)}{\partial d_i} \right)^{-1} \right)^{-1} =: \eta(d) \end{aligned}$$

We will show that if an allocation $d \in \mathcal{C}(v)$ is the limit of a sequence of stochastically stable allocations as $\delta \to 0$, then d must maximize $\eta(d)$ over the domain of core allocations. The proof is by contradiction. Let d^* be a limit of a sequence of stochastically stable allocations as $\delta \to 0$. If d^* does not maximize $\eta(d)$ then there must exist $\tilde{d} \in \mathcal{C}(v)$ such that $\eta(\tilde{d}) > \eta(d^*)$. Let $d^{i*}, \delta^i, i = 1, \ldots, \infty$, be a sequence of stochastically stable allocations converging to d^* and their corresponding values of δ. Choose $\hat{d} \in \mathcal{C}^*(v)$ close to $\tilde{d}$ so that $\eta(d^*) < \eta(\hat{d})$ and let $\hat{d}^i, \delta^i$, $i = 1, \ldots, \infty$, be a sequence of allocations converging to $\hat{d}$, corresponding to the same δ^i as in the previous sequence.

$$\begin{aligned} &\xi(d^{i*}, \delta^i) \to \eta(d^*) \text{ and } \xi(\hat{d}^i, \delta^i) \to \eta(\hat{d}) \\ &\quad \Longrightarrow \quad \exists N_1 \text{ such that } \forall i \geqslant N_1\text{:} \quad \xi(\hat{d}^i, \delta^i) > \xi(d^{i*}, \delta^i) \end{aligned}$$

Now by Lemma 1:

$$\exists\hat{\delta} \text{ such that } \forall\delta < \hat{\delta}: \quad \beta(\hat{d}) = \mu^*(\hat{d})$$

As $\hat{d} > 0$ and the bounds in Lemma 1 are continuous in d:

$$\exists\hat{\hat{\delta}},\ \epsilon > 0 \text{ such that } \forall\delta < \hat{\hat{\delta}}: \quad \|d - \hat{d}\| \leqslant \epsilon \quad \Longrightarrow \quad \beta(d) = \mu^*(d)$$

Take one such $\hat{\hat{\delta}}, \epsilon$ pair and let $N_2 = \inf\{i\colon \delta^i < \hat{\hat{\delta}},\ \|\hat{d}^i - \hat{d}\| \leqslant \epsilon\}$. Let $N = \max\{N_1, N_2\}$. Then $\forall i > N$:

$$\mu^*(\hat{d}^i) > \mu^*(d^{i*}) \quad \text{and} \quad \beta(\hat{d}^i) = \mu^*(\hat{d}^i)$$

which implies d^{i*} is not stochastically stable by Lemma 3. Contradiction.

So if d^* is the limit of a sequence of stochastically stable allocations as $\delta \to 0$, then d^* solves:

$$\max_{d\in\mathcal{C}(v)} \eta(d) = \max_{d\in\mathcal{C}(v)} \min_{j\in I} \left(\sum_{i\neq j} \left(\alpha_i \frac{\partial \log u_i(d_i)}{\partial d_i} \right)^{-1} \right)^{-1}$$
$$= \max_{d\in\mathcal{C}(v)} \left(\max_{j\in I} \sum_{i\neq j} \left(\alpha_i \frac{\partial \log u_i(d_i)}{\partial d_i} \right)^{-1} \right)^{-1}$$

So d^* must solve:

$$\min_{d\in\mathcal{C}(v)} \max_{j\in I} \sum_{i\neq j} \left(\alpha_i \frac{\partial \log u_i(d_i)}{\partial d_i} \right)^{-1}$$

Now:

$$\left(\frac{\partial \log u_i(d_i)}{\partial d_i} \right)^{-1} = u_i(d_i) \left(\frac{\partial u_i(d_i)}{\partial d_i} \right)^{-1} = \frac{u_i(d_i)}{u_i'(d_i)}$$

So if $\frac{u_i(d_i)}{u_i'(d_i)}$ is strictly convex then as a weighted sum of strictly convex functions:

$$\sum_{i\neq j} \frac{1}{\alpha_i} \left(\frac{\partial \log u_i(d_i)}{\partial d_i} \right)^{-1}$$

is strictly convex $\forall j$ and:

$$\max_{j\in I} \sum_{i\neq j} \frac{1}{\alpha_i} \left(\frac{\partial \log u_i(d_i)}{\partial d_i} \right)^{-1}$$

being the maximum of strictly convex functions is also strictly convex. It follows that our minimization problem, being one of minimizing a strictly convex function over a convex set, has a unique solution. □

References

[1] J.R. Green, The stability of edgeworth's recontracting process, Econometrica 42 (1974) 21–34.
[2] A.M. Feldman, Recontracting stability, Econometrica 42 (1974) 35–44.
[3] M. Agastya, Adaptive play in multiplayer bargaining situations, Rev. Econ. Stud. 64 (1997) 411–426.
[4] M. Agastya, Perturbed adaptive dynamics in coalition form games, J. Econ. Theory 89 (1999) 207–233.
[5] K. Rozen, Conflict leads to cooperation in Nash bargaining, Working Papers 39, Yale University, Department of Economics, 2008.

[6] D. Foster, H.P. Young, Stochastic evolutionary game dynamics, Theoret. Population Biol. 38 (1990) 219–232.
[7] M. Kandori, G.J. Mailath, R. Rob, Learning, mutation, and long run equilibria in games, Econometrica 61 (1993) 29–56.
[8] K. Chatterjee, B. Dutta, D. Ray, K. Sengupta, A noncooperative theory of coalitional bargaining, Rev. Econ. Stud. 60 (1993) 463–477.
[9] M. Perry, P.J. Reny, A noncooperative view of coalition formation and the core, Econometrica 62 (1994) 795–817.
[10] H.P. Young, An evolutionary model of bargaining, J. Econ. Theory 59 (1993) 145–168.
[11] R. Serrano, O. Volij, Mistakes in cooperation: the stochastic stability of edgeworth's recontracting, Econ. J. 118 (2008) 1719–1741.
[12] T. Arnold, U. Schwalbe, Dynamic coalition formation and the core, J. Econ. Behav. Organ. 49 (2002) 363–380.
[13] H.P. Young, The evolution of conventions, Econometrica 61 (1993) 57–84.
[14] H.P. Young, Individual Strategy and Social Structure, Princeton University Press, 1998.