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The core of roommate problems: size and rank-fairness within matched pairs. (English summary)

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Consider a roommate problem consisting of a finite set of *players* N and a *ranking* $r_i\colon N\to N$ for each $i\in N$ so that $r_i(j)$ is the rank of player j in player i's ranking. It is assumed that $r_i(i)=|N|$ for all $i\in N$. The rankings represent ordinal preferences over partners, so that $r_i(j)=1$ means that j is player i's most preferred partner, $r_i(j)=2$ means that j is player i's second most preferred partner, and so on.

The *disagreement* over player i is $\Delta_i=\max_{j\in N\setminus i} r_j(i)-\min_{j\in N\setminus i} r_j(i)$. The *maximal disagreement* is $\Delta=\max_{i\in N}\Delta_i$.

A *matching* is a function $\mu\colon N\to N$ such that $\mu(\mu(i))=i$ for all $i\in N$. The *partner* of i at matching μ is $\mu(i)$. A matching μ is *stable* if there do not exist two players such that both would rather be matched to one another than to their partners at μ.

The *rank gap* of i and $j=\mu(i)$ at μ is given by $|r_i(j)-r_j(i)|$. The *maximal rank gap* at μ is the maximum of this quantity over all $i,j\in N$ such that $j=\mu(i)$.

Theorem 1 of this paper bounds the maximal rank gap at any stable matching in terms of the maximal disagreement. The proof for $n\ge 6$ is as follows (the proof for $n<6$ is more simple).

Take a given μ and $i\in N$. Let $\mu(i)=j$. There are two cases to consider.

Case 1. If $r_i(j)=r_j(i)$, then the rank gap of i and j is zero.

Case 2. If $r_i(j)\neq r_j(i)$, then either $r_i(j)\neq 1$ or $r_j(i)\neq 1$. Assume without loss of generality that $r_i(j)\neq 1$.

Let J' be the set of players that i prefers to j. Let $j'\in J'$ and let $i'=\mu(j')$. It must be that j' prefers i' to i or else μ would not be stable. That is, $r_{j'}(i')<r_{j'}(i)$; hence

$$r_{j'}(i')-r_{j'}(i)<0. \tag{1}$$

We then have

$$r_j(i')-r_j(i)=\underbrace{r_j(i')-r_{j'}(i')}_{\le\Delta_{i'}\le\Delta}+\underbrace{r_{j'}(i')-r_{j'}(i)}_{<0\text{ by (1)}}+\underbrace{r_{j'}(i)-r_j(i)}_{\le\Delta_i\le\Delta}<2\Delta.$$

Hence, $r_j(i')\le r_j(i)+2\Delta-1$ so that i' is among the $r_j(i)+2\Delta-1$ most preferred partners of player j. This holds for every $i'=\mu(j')$, $j'\in J'$. The set of such players has size $|J'|=r_i(j)-1$. As $\Delta>0$, we also have $r_j(i)\le r_j(i)+2\Delta-1$, so that i is also among the $r_j(i)+2\Delta-1$ most preferred partners of player j. Therefore, in total we have at least $r_i(j)-1+1=r_i(j)$ players among the $r_j(i)+2\Delta-1$ most preferred partners of player j. That is, $r_i(j)\le r_j(i)+2\Delta-1$. This implies $|r_i(j)-r_j(i)|\le 2\Delta-1$. That is, the rank gap of i and j is no greater than $2\Delta-1$. This completes the proof.

The authors go on to show the tightness of their bounds by means of some intricately constructed examples. They also give bounds for average (rather than maximal) rank gaps in terms of average (rather than maximal) disagreements. *Jonathan Newton*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.