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The hat guessing number of graphs. (English summary)
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The *hat guessing game* involves n players who are identified with the vertices of a graph. Each player wears a hat which is one of q possible colors. Players can observe the colors of the hats worn by their neighbors on the graph but cannot observe the color of their own hat. All of the players simultaneously guess their own hat color according to a predetermined strategy which can depend on the colors of the hats worn by their neighbors. The *hat guessing number* $HG(G)$ of a graph G is the largest integer q such that there exists a guessing strategy which guarantees that at least one player guesses correctly no matter which colors are assigned.

Consider the complete bipartite graph $K_{n,n}$, consisting of two sets, V_L and V_R, of n vertices each, with each vertex adjacent to every member of the other set. The paper answers the conjecture of whether there exists a constant $\alpha > 0$, independent of n, such that $HG(K_{n,n}) \geq n^\alpha$ for sufficiently large n. Specifically, it is shown that $HG(K_{n,n}) \geq n^{\frac{1}{2}-o(1)}$.

The proof is constructed as follows. Lemma 3.1 shows that $HG(K_{m,n}) \geq q$ if there is a guessing strategy such that, for any given coloring y of V_R, the set of colorings $\mathcal{C}_y$ of V_L such that none of the players in V_R choose the correct color must be contained within a *Hamming ball* of radius $m-1$ centered on some coloring of V_L. If all m players in V_L then guess according to this latter coloring, then at least one of them must guess correctly. Lemma 3.2 then observes that any set of at most m colorings of V_L must be contained in some such Hamming ball.

The proof proceeds (Lemma 4.1) to consider a certain type of matrix whose rows are indexed by the n vertices in V_R and whose columns are indexed by the q^m different possible colorings of V_L. The entries specify color choices for vertices in V_R given the colorings of V_L. This matrix is *saturated* in that if we select a set of $m+1$ different colorings for V_L, then the choices of vertices in V_R span all possible colors. Therefore, any given set $\mathcal{C}_y$, as defined above, must contain no more than m elements, or else it would contradict its definition. Therefore, if a saturated matrix exists, then by Lemmas 3.1 and 3.2 we have $HG(K_{m,n}) \geq q$.

The final part of the proof (Proposition 4.3) substitutes conditions (Lemma 3.4) under which a saturated matrix exists. This implies that $HG(K_{n,n}) \geq n^{\frac{1}{2}-o(1)}$.

In addition to the question discussed above, the paper also considers multipartite graphs, oriented graphs and linear guessing strategies. *Jonathan Newton*

References

1. G. Aggarwal, A. Fiat, A. V. Goldberg, J.D. Hartline, N. Immorlica, M. Sudan, Derandomization of auctions, in: Proceedings of the Thirty-Seventh Annual ACM Symposium on Theory of Computing, STOC'05, 2005, pp. 619–625. MR2181666
2. N. Alon, Combinatorial Nullstellensatz, Comb. Probab. Comput. 8 (1–2) (1999) 7–29. Recent trends in combinatorics (Mátraháza, 1995). MR1684621
3. N. Alon, O. Ben-Eliezer, C. Shangguan, I. Tamo, The hat guessing number of graphs,

in: 2019 IEEE International Symposium on Information Theory, ISIT, IEEE, 2019, pp. 490–494.
4. N. Alon, K. Efremenko, B. Sudakov, Testing equality in communication graphs, IEEE Trans. Inf. Theory 63 (11) (2017) 7569–7574. MR3724444
5. N. Alon, J.H. Spencer, The Probabilistic Method, fourth edition, Wiley Series in Discrete Mathematics and Optimization, John Wiley & Sons, Inc., Hoboken, NJ, 2016. MR3524748
6. N. Alon, M. Tarsi, A nowhere-zero point in linear mappings, Combinatorica 9 (4) (1989) 393–395. MR1054015
7. Z. Bar-Yossef, Y. Birk, T.S. Jayram, T. Kol, Index coding with side information, in: 2006 47th Annual IEEE Symposium on Foundations of Computer Science, FOCS'06, Oct 2006, pp. 197–206. MR2815830
8. S. Bhandari, J. Radhakrishnan, Bounds on the zero-error list-decoding capacity of the $q/(q-1)$ channel, arXiv preprint, arXiv:1802.08396, 2018.
9. S. Butler, M.T. Hajiaghayi, R.D. Kleinberg, T. Leighton, Hat guessing games, SIAM J. Discrete Math. 22 (2) (2008) 592–605. MR2399367
10. S. Chakraborty, J. Radhakrishnan, N. Raghunathan, P. Sasatte, Zero error list-decoding capacity of the $q/(q-1)$ channel, in: FSTTCS 2006: Foundations of Software Technology and Theoretical Computer Science, in: Lecture Notes in Comput. Sci., vol. 4337, Springer, Berlin, 2006, pp. 129–138. MR2335328
11. O. Chervak, Warwick combinatorics seminar, https://warwick.ac.uk/fac/sci/maths/research/events/seminars/areas/combinatorics/2016-17.
12. T. Ebert, Applications of recursive operators to randomness and complexity, PhD thesis, University of California, Santa Barbara, 1998. MR2698364
13. T. Ebert, W. Merkle, H. Vollmer, On the autoreducibility of random sequences, SIAM J. Comput. 32 (6) (2003) 1542–1569. MR2034250
14. P. Elias, Zero error capacity under list decoding, IEEE Trans. Inf. Theory 34 (5, part 1) (1988) 1070–1074. MR0982818
15. P. Erdős, L. Lovász, Problems and results on 3-chromatic hypergraphs and some related questions, Colloq. Math. Soc. János Bolyai 10 (1975) 609–627. MR0382050
16. M. Farnik, A hat guessing game, PhD thesis, Jagiellonian University, 2015.
17. U. Feige, You can leave your hat on (if you guess its color), Technical Report MCS04-03, The Weizmann Institute of Science, 2004.
18. M.L. Fredman, J. Komlós, On the size of separating systems and families of perfect hash functions, SIAM J. Algebraic Discrete Methods 5 (1) (1984) 61–68. MR0731857
19. M. Gadouleau, Finite dynamical systems, hat games, and coding theory, SIAM J. Discrete Math. 32 (3) (2018) 1922–1945. MR3835237
20. M. Gadouleau, N. Georgiou, New constructions and bounds for Winkler's hat game, SIAM J. Discrete Math. 29 (2) (2015) 823–834. MR3337992
21. M. Gadouleau, S. Riis, Graph-theoretical constructions for graph entropy and network coding based communications, IEEE Trans. Inf. Theory 57 (10) (2011) 6703–6717. MR2882254
22. W. Haemers, On some problems of Lovász concerning the Shannon capacity of a graph, IEEE Trans. Inf. Theory 25 (2) (1979) 231–232. MR0521317
23. W. Haemers, An upper bound for the Shannon capacity of a graph, in: Algebraic Methods in Graph Theory, vol. I, II, Szeged, 1978, in: Colloq. Math. Soc. János Bolyai, vol. 25, North-Holland, Amsterdam-New York, 1981, pp. 267–272. MR0642046
24. J. Körner, Fredman-Komlós bounds and information theory, SIAM J. Algebraic Discrete Methods 7 (4) (1986) 560–570. MR0857591
25. M. Krzywkowski, A modified hat problem, Comment. Math. 50 (2) (2010) 121–126.

MR2789280
26. M.P. Krzywkowski, Hat problem on a graph, PhD thesis, University of Exeter, 2012. MR2674427
27. E. Lubetzky, U. Stav, Nonlinear index coding outperforming the linear optimum, IEEE Trans. Inf. Theory 55 (8) (2009) 3544–3551. MR2598057
28. R. Peeters, Orthogonal representations over finite fields and the chromatic number of graphs, Combinatorica 16 (3) (1996) 417–431. MR1417351
29. S. Riis, Information flows, graphs and their guessing numbers, Electron. J. Comb. 14 (2007) R44. MR2320600
30. S. Robinson, Why mathematicians now care about their hat color, N.Y. Times 10 (April 2001) D5.
31. C.E. Shannon, The zero error capacity of a noisy channel, IRE Trans. Inf. Theory IT-2 (September) (1956) 8–19. MR0089131
32. J.B. Shearer, On a problem of Spencer, Combinatorica 5 (3) (1985) 241–245. MR0837067
33. W. Szczechla, The three colour hat guessing game on cycle graphs, Electron. J. Comb. 24 (1) (2017) P1.37. MR3625914
34. P. Winkler, Games people don't play, in: D. Wolfe, T. Rodgers (Eds.), Puzzlers' Tribute: A Feast for the Mind, A K Peters, Natick, MA, 2002, pp. 301–313. MR2034896

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.