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Two problems in max-size popular matchings. **(English summary)**
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Let the graph $G=(V,E)$ comprise a set of agents V and a set of edges E which contains unordered pairs of distinct elements of V. If $(u,u')\in E$, we say that u and u' are *neighbors*. An edge $(u,u')\in E$ indicates that $u,u'\in V$ are acceptable matches for one another. Each $u\in V$ is associated with a capacity $\mathbf{cap}(u)\geq 1$ and a strict ranking $\succ_u$ over the neighbors of u in G.

A *matching* M in G is a subset of E such that $|M(u)|\leq \mathbf{cap}(u)$ for all $u\in V$, where $M(u)$ is the set of neighbors of u in the graph $G'=(V,M)$. A *max-size* matching is a matching with the greatest number of edges of any matching.

Consider the following voting procedure between two matchings M and M'. For any given $u\in V$, consider the sets $M(u)$ and $M'(u)$. Add null agents to the smaller of the two sets to make the sets the same size. These null agents are always less preferred than any other agent. From each of the resulting sets, remove any agents that are in both sets. Let the resulting (non-intersecting, similarly sized) sets be $\overline{M}(u)$ and $\overline{M}'(u)$. Consider a bijection between $\overline{M}(u)$ and $\overline{M}'(u)$. For each agent $v\in\overline{M}(u)$, we have a *win* if u prefers v to the image of v in $\overline{M}'(u)$. We have a *loss* if the reverse holds. The voting score of u for this bijection equals the number of wins minus the number of losses. The voting score of u for M versus M' minimizes the voting score across all possible bijections.

The popularity score of M versus M' is the sum of voting scores over all $u\in V$. We say that M is *at least as popular* as M' if the popularity score of M versus M' is nonnegative. A matching M is a *popular matching* if it is at least as popular as every matching in G. A *max-size popular matching* is a matching with the greatest number of edges of any popular matching.

The paper under review first considers the case in which G is bipartite (e.g., student to college matchings). A "2-level" version of the Gale and Shapley algorithm [D. Gale and L. S. Shapley, Amer. Math. Monthly **69** (1962), no. 1, 9–15; MR1531503] is given that finds a max-size popular matching. The running time of the algorithm is linear in $|E|+|V|$. Furthermore, it transpires that every pairwise stable matching is popular. Hence, the authors describe popularity as a weaker notion of stability than pairwise stability. Furthermore, max-size popularity is a global condition in that it depends on maximizing a global quantity (for more on local and global stability conditions, see [J. Newton and R. Sawa, J. Econom. Theory **157** (2015), 1–27; MR3335933]).

When G is not bipartite, even if every agent has a capacity of one (the roommate problem), a popular matching does not necessarily exist. Furthermore, even when a popular matching exists, finding a max-size popular matching is an NP-hard problem.

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.