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Connector-breaker games on random boards. (English summary)

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The $(m:b)$ *maker-breaker connectivity game* is played on a graph G. First, the *maker* claims up to m edges of the graph. Following this, the *breaker* claims up to b edges. The players continue to alternate turns. The maker wins if she manages to claim edges that are a spanning tree of G. Otherwise, the breaker wins.

The $(m:b)$ *connector-breaker connectivity game* is a variant of the maker-breaker game in which the maker needs to choose her edges in such a way that the graph consisting of her claimed edges stays connected throughout the game.

If both players play optimally, then for given m, b, G, either the maker will win or the breaker will win. For a random graph $G \sim G_{n,p}$ in which each edge appears with probability p, there exists a threshold p^* such that when $p < p^*$, the breaker wins asymptotically (as $n \to \infty$) almost surely, and when $p > p^*$, the maker wins asymptotically almost surely.

In the maker-breaker game there is a close connection between (i) threshold levels of b above which the breaker wins when G is the complete graph, and (ii) the threshold p^* for $G \sim G_{n,p}$. Specifically, the thresholds are proportional to $\frac{n}{\log n}$ and $\frac{\log n}{n}$, respectively.

For the $(2:b)$ connector-breaker game on the complete graph, thresholds on b are also proportional to $\frac{n}{\log n}$. Thus, it is reasonable to conjecture that for the $(2:2)$ connector-breaker game with $G \sim G_{n,p}$, the threshold p^* would be proportional to $\frac{\log n}{n}$. The main result of the paper, Theorem 1.3, shows that this is not true and that p^* is of size $n^{-\frac{2}{3}+o(1)}$.

The proof shows that the maker, if she wishes to reach a vertex x from another vertex r, can do so if the graph contains a *good structure*, a full binary tree with root r and k levels, such that every leaf of the tree is adjacent to x. As k tends to infinity, the density of this structure tends to $\frac{3}{2}$, hence many such structures appear when $p > n^{-\frac{2}{3}}$, helping maker to win the game. Conversely, when $p < n^{-\frac{2}{3}}$, the lack of good structures helps breaker to win. *Jonathan Newton*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.