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Games on signed graphs. (English summary)
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The power allocation game is defined as follows. Consider a set of n *countries*. Each country has a subset of countries which are friends and a (possibly empty) subset of countries which are adversaries. Each country is a friend of itself. If i is a friend (respectively, enemy) of j, then j is a friend (respectively, adversary) of i.

Each country has a given endowment of *power*. A strategy for a country allocates power to its friends and adversaries. The total allocation by a country equals its endowment. Allocating to a friend is interpreted as helping a friend. Allocating to an adversary is interpreted as supporting the demise of the adversary.

If the total amount of power allocated to helping a country strictly exceeds the power allocated to supporting its demise, the country is said to be *safe*. If the total amount of power allocated to helping a country is strictly lower than the power allocated to supporting its demise, the country is said to be *unsafe*. If the amounts are equal, the country is said to be *precarious*.

It is shown (Theorem 1) that there exists a utility function that satisfies some preference axioms. These axioms are that (1) a strategy profile V is weakly preferred to a strategy profile U if the set of friends who are safe or precarious at V includes the set of friends who are safe or precarious at U, and the set of adversaries who are unsafe or precarious at V includes the set of adversaries who are unsafe or precarious at U; (2) a country is indifferent between two strategies if the outcome (safe/unsafe/precarious) for all of the country's friends and enemies is the same at each of these strategies; (3) V is strictly preferred to U by a country, if the country is safe or precarious under V, but unsafe under U.

Theorem 2 shows that a pure strategy Nash equilibrium of this game exists. The proof proceeds by showing (Lemma 1) that a pure Nash equilibrium exists if the vector of countries' powers can be written as an affine decomposition $p = Bd + c$ in which (1) B is an n by q incidence matrix that tracks whether or not each country is party to each of the q adversarial relationships in the game, (2) d is a q by 1 non-negative vector and c is a n by 1 non-negative vector, (3) if countries i and j are adversaries, then it is not the case that c_i and c_j are both strictly positive. It is then shown that an algorithm can be used to construct such a decomposition.

The paper proceeds to further explore the properties of this game before defining and studying an additional game in which countries choose friendly or adversarial relationships.

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