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Maker-breaker percolation games II: escaping to infinity. (English summary)

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Let Λ be an infinite connected (multi)graph, and let v_0 be a vertex of Λ. In the (p, q)-percolation game on (Λ, v_0), two players, Maker and Breaker, play in alternating turns, with Maker playing first. Initially, all edges of Λ are marked as unsafe. On each of her turns, Maker marks p unsafe edges as safe, while on each of his turns Breaker takes q unsafe edges and deletes them from the graph. Breaker wins if, at any time in the game, the component containing v_0 becomes finite. Conversely, if Maker is able to ensure that v_0 remains in an infinite component indefinitely, then we say she has a winning strategy.

Consider the case in which Λ is the 2-dimensional integer lattice on vertex set $\mathbb{Z}^2$ whose edges consist of pairs of vertices $v, w \in \mathbb{Z}^2$ with Euclidean distance $\|v - w\| = 1$. On such a graph, v_0 can be arbitrary. The main theorems of the paper are as follows.

Theorem 1.3 shows that if $p \geq 2q$, then Maker has a winning strategy. Theorem 1.4 shows that if $q \geq 2p$, then Breaker has a winning strategy.

The proof of Theorem 1.3 builds upon Part I [Combin. Probab. Comput. **30** (2021), no. 2, 200–227; MR4225784]. In Part I, a game was played on $\mathbb{Z} \times P_n$, the subgraph of $\mathbb{Z}^2$ induced by the vertex set $\{(x, y) : x \in \mathbb{Z}, y \in [n]\}$. In that game, two players took turns claiming vertices. It was shown that if a player can claim twice as many edges on his turn as his opponent, then, for large enough n, he can prevent the other player from ever claiming edges that constitute a vertical cut through $\mathbb{Z} \times P_n$. In the context of Theorem 1.3, this result is used to show that there is no way for the Breaker to delete edges fast enough to trap v_0 in a finite component of the graph.

The proof of Theorem 1.4 surrounds v_0 with square annuli. For the Maker to win, she must be able to escape (by denoting safe edges) from any of these annuli. A similar argument to the proof of Theorem 1.3 is used to show that the Maker cannot escape being trapped by the Breaker on any given side (i.e., to the top, bottom, left or right). It remains to show that the Maker cannot escape via the corners of annuli. However, unlike the sides, the size of the corners does not increase with the size of the annulus. Therefore, it is possible to use results from another game in the literature, the box-game, to show that the Breaker can delete edges in the corners fast enough to prevent the Maker escaping.

{For Part I see MR4225784.}

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.