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Peg solitaire on Cartesian products of graphs. **(English summary)**
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The following describes *peg solitaire*. Consider a connected, undirected graph $G = (V, E)$. Put pegs in all of the vertices except one. This is the *starting state*. From this state, a process is followed. At each step in the process, for some set of three vertices u, v, w such that u is adjacent to v, v is adjacent to w, u and v have pegs and w has no peg, move the peg from u to w, removing and discarding the peg at v which has been "jumped over". A *terminal state* is a state such that no more jumps are possible.

A graph G is *solvable* if there is some starting state such that a terminal state with only a single peg can be reached. G is *freely solvable* if from every starting state, a terminal state with only a single peg can be reached. G is *super freely solvable* if from any starting state, for any target vertex, a terminal state with only a single peg at the target vertex can be reached.

The Cartesian product of two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$, denoted $G \square H$, has vertex set $V_G \times V_H$ and edges between any $(v_G, v_H), (v'_G, v'_H) \in V_G \times V_H$ such that either $v_G = v'_G$ and there is an edge between v_H and v'_H in H, or $v_H = v'_H$ and there is an edge between v_G and v'_G in G.

The main theorem of the paper (Theorem 4) is that if G has an even Hamiltonian path and H is connected, then $G \square H$ is solvable. The proof proceeds as follows.

Let P_n be a path with n vertices. Theorem 1 proves, by construction, that $P_n \square P_2$, $n \geq 3$, is super freely solvable. This result can be extended by induction to all $P_m \square P_n$, $m \geq 3$ or $n \geq 3$ (Theorem 2).

Theorem 3 states that $P_2 \square G$ is freely solvable for any connected G. The proof takes a spanning tree of G rooted at arbitrary root vertex v_0, decomposes the spanning tree into paths, then considers the Cartesian product of each of these paths with P_2, using Theorem 1 at each step.

Finally, Theorem 4 is proven by considering the Hamiltonian cycle $\{v_1, v_2, v_3, \ldots\}$ in G two vertices at a time. Assume, without loss of generality, that the starting state has no peg at (v_1, w) for some $w \in H$. Theorem 3 can be applied to $\{v_1, v_2\} \square H$. It is then shown that, whichever state is arrived at from applying Theorem 3, it takes two steps to reach a state such that no pegs remain in $\{v_1, v_2\} \square H$ and there is a vertex without a peg in $\{v_3, v_4\} \square H$. Iteratively apply Theorem 3 to successive pairs of vertices in the Hamiltonian cycle. As the cycle is even, this process eventually terminates with one remaining peg.

Aside from the main theorem, the paper also addresses other interesting topics, notably showing that the Cartesian product of any two stars is solvable (Theorem 5).

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.